

**Invariance of Positive-Frequency Kernels
in Generalized FRW Spacetimes**

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We consider the Klein-Gordon equation in FRW-like spacetimes, with compact space sections (not necessarily isotropic neither homogeneous). The bi-scalar kernel allowing to select the positive-frequency part of any solution is developed on mode solutions, using the eigenfunctions of the three-dimensional Laplacian. Of course, this kernel is not unique but, except (perhaps) when the scale factor undergoes a special law of evolution, the metric has no more symmetries (connected with the identity) than those inherited from the space sections. As a result, all admissible definitions of the positive-frequency kernel are related one to another by a unitary transformation which commutes with the connected isometries of spacetime; any such kernel is invariant under these isometries. A physical interpretation is tentatively suggested.

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I. INTRODUCTION.

Having in mind a theory of *free* quantum fields in a Lorentzian manifold V_4 , we plan to later undertake the construction of Fock space. But the first step will be to consider one-particle states; therefore we write the wave equation describing the minimal coupling of a particle with mass m

$$(\nabla^2 + m^2)\Psi = 0 \tag{1}$$

for a complex-valued function Ψ of class C^∞ . The theory of its elementar solutions, satisfying retarded or advanced conditions, has been intensively developed [1][2]. For a given manifold they are unique and invariant by the isometries of V_4 [2]. In the case of Friedman-Robertson-Walker (FRW) models, explicit expressions for these propagators have been calculated [3][5][4].

Of different nature are the kernels associated with the selection of a positive-frequency part in the solutions [6]. In spite of being related to the propagator by a convolution equation [7] they cannot be just derived from the fundamental solutions. Worse than that, their definition seems to suffer (except in the stationary case) from a high degree of arbitrariness. Insofar as FRW universes are concerned, it is in principle possible to calculate the G_1 kernel of Lichnerowicz along the lines of ref [4]. But the mathematical framework used in that work, justified if one aims at a general theory, is unnecessarily complicated for applications to FRW spacetimes, mainly because it disregards the concept of mode solutions. Therefore obtaining compact and tractable formula will be a new advance. In addition we shall be concerned by the question of symmetries.

We are faced with the problem of splitting the linear space of the solutions to equation (1) into some generalization of positive-frequency and negative-frequency subspaces [8]. In a popular terminology one speaks of the problem of "defining the vacuum". In our opinion it is rather the problem of defining one-particle states.

The determination of a positive-frequency subspace amounts to the selection of a "complex-

structure positive operator” [9][7][4], that is a real linear operator J (acting on solutions) such that $J^2 = -1$. The action of J can be expressed with the help of a kernel G_1 in terms of the sesquilinear form associated with the Gordon current [7]. This kernel can be uniquely selected in the particular case of stationary spacetime [10]; in this case energy (defined as time component of momentum) is conserved, and positive frequency corresponds to positive energy. In contradistinction, quantization in *nonstationary spacetimes* is generally plagued with ambiguities about the determination of positive-frequency solutions.

The lack of unicity in the splitting of the space of solutions is somehow redeemed by the well-known fact that two possible candidates as positive-frequency subspace can be mapped one onto the other by a unitary transformation. Known for a long time in the FRW case, this property has been extended to arbitrary spacetime by the work of Moreno [11]. This wide class of equivalence can be further restricted if we introduce reasonable symmetry requirements. We mean for instance that quantum field theory in any spacetime should respect isometries, as much as it does in the Minkowski case. In fact a similar concern has been already expressed long time ago in the special case of de Sitter space [12].

Notice however that, in special relativity, only orthochronous Lorentz transformations preserve positivity of energy. Similarly, in curved spacetime, the role of discrete isometries should be investigated; but for simplicity of the exposition, we shall mainly focus on the isometries that are continuously connected with the identity (*connected isometry group*). The possibility of including discrete isometries will be briefly discussed at the end of this work.

As most metrics of physical interest enjoy some symmetries, we shall complete the axiomatics of ref. [7] by a condition of *isometric invariance*.

Technically, we prefer to deal with the positive-frequency and negative-frequency projectors $\Pi^\pm = \frac{1 \pm iJ}{2}$ rather than with J . This notation amounts to consider the positive-energy and negative-energy kernels $D^\pm(x, y)$ instead of G_1 . They must be distribution solutions

to the wave equation (1) and satisfy the formula

$$(\Psi^\pm)(y) = (D_y^\pm; \Psi) \quad (2)$$

where $\Psi^\pm = \Pi^\pm \Psi$ is the positive-frequency part of Ψ , y is an arbitrary point of V_4 , and $(\Phi; \Psi)$ denotes the sesquilinear form constructed with the help of the Gordon current, conservative provided Φ and Ψ are solutions to the KG equation. Our interest for this approach is motivated by the fact that the field operator must be defined through the creation and annihilation operators associated with the one-particle state D^+ . In terms of this kernel, we shall formulate isometric invariance as follows

$$D^+(x, y) = D^+(Tx, Ty) \quad (3)$$

for any metric-preserving transformation T of the connected isometry group. When this property is satisfied one can reasonably expect that field operators, commutation relations, and the whole machinery of Fock space construction will be isometrically invariant. Of course, the condition of isometric invariance substantially reduces the arbitrariness in the choice of J , without fixing it completely.

Our purpose is now to prove that, for a large class of nonstationary spacetimes, all reasonable splittings of the space of solutions are mutually equivalent, *in a way which respects the symmetries of free motion*, and more specially isometric invariance.

In this paper we shall consider a simple form of nonstatic orthogonal spacetime, large enough to encompass ordinary FRW universes (without singularity) as a particular case. In analogy with usual FRW universes, the spacetimes we deal with admit, in the generic case, distinguished timelike curves and space sections. In order to obtain rigorous results we shall eventually assume that V_4 can be identified with $\mathbf{R} \times V_3$ where V_3 is a connected, compact, three-dimensional space. In contrast to the usual FRW models, neither isotropy nor homogeneity of space sections is necessary here.

All the space of the form we consider have interesting features:

- i)* The wave equation can be reduced to an ordinary (second order) differential equation, complemented by the eigenvalue equation for the three-dimensional Laplacian. This reduction stems from the conservation of a (purely spatial) generalization of kinetic energy.
- ii)* At least in the generic case, the isometry group is under control.

For spacetimes of this kind, the dynamical symmetry group contains not only isometries but also the transformation generated by "kinetic energy".

The main achievement of this article states that the demand of isometric invariance, associated with the natural requirement that the separation of frequencies respects kinetic-energy shells, permits us to select a unique kernel D^+ , *up to a unitary transformation commuting with all connected isometries*. This result is in the line of the main theorem proved in ref.[4]; but the work presented here goes one step further by taking the role of symmetries into account. In [4] explicit formulas concerning the Lichnerowicz kernel were displayed. Here, the expansion we obtain directly concerns the positive-frequency kernels D^+ . Strictly speaking it is not a closed form expression, but it is reasonably compact and its invariance properties can be easily read off.

Notice also that Moreno in ref.[4] treated the Klein-Gordon equation as an evolution equation, and this was probably a technical necessity for his purpose. Here we avoid this complication, unnecessary in the simple case of FRW-like spacetimes.

in Section II, the intuitive feeling that "in general" FRW spacetimes have no more symmetries than those provided by the group of motion in space sections is confirmed and given a precise meaning.

We write down in Section III, for the D^+ kernels, a tractable expansion over modes and over the eigenfunctions of the three-dimensional Laplacian. We explicitly check that this expression is invariant under all the isometries of the space sections.

Finally the unitary equivalence of all admissible definitions of D^+ will be exhibited, in a way which respects all the isometries of V_3 , ensuring by the same token (in the generic

case) invariance under all the transformations of the *connected* isometry group of V_4 .
Section IV is devoted to a few comments and a tentative interpretation.

II. GENERALIZED FRW-SPACETIMES

II.1 Generalized kinetic energy

We assume that the spacetime manifold is $V_4 = \mathbf{R} \times V_3$ and for some t

$$ds^2 = B^6 dt^2 - B^2 d\sigma^2 \quad (4)$$

where B is a strictly positive function of $x^0 = t$ and $d\sigma^2 = \gamma_{ij}(x^k)dx^i dx^j$ defines an elliptic metric. Notice that $(V_3, d\sigma^2)$ has arbitrary curvature.

We could equivalently write $ds^2 = d\bar{t}^2 - B^2 d\sigma^2$, if we were to use the cosmic time \bar{t} .

However, unless otherwise specified, we shall stick to the form (4) of the metric because it eliminates first derivatives from the reduced wave equation and leads to a simple formula for the Gordon current, implying equation (12) below.

In spacetimes of the above type, the dynamics of a free particle enjoys an interesting property. Indeed *free motion in these spacetimes admits a first integral which generalizes kinetic energy* [13].

This can be checked already at the *classical* level using a covariant symplectic Hamiltonian framework. In this formalism, phase space is the cotangent bundle $T_*(V_4)$ endowed with coordinates x^α, p_β submitted to the standard Poisson bracket relations.

Geodesic motion is generated by the Hamiltonian function

$$H = \frac{1}{2} g^{\alpha\beta} p_\alpha p_\beta$$

Taking (4) into account and defining

$$2K = \gamma^{ij} p_i p_j$$

where γ with latin superscripts refers to the inverse tensor of γ_{ij} , say $\gamma^{ij}\gamma_{jk} = \delta_k^i$, we easily obtain

$$2H = B^{-6} p_0 p_0 - 2B^{-2} K$$

Since $\{x^i, p_0\} = \{x^0, p_j\} = 0$ it follows that K has a vanishing Poisson bracket with $B^{-6}p_0p_0$ and B^{-2} hence finally also with H , which proves our statement. Notice that K involves space variables x^k, p_j only.

The *quantum* mechanical version of K is proportional to the three-dimensional Laplacian. Its commutation with the Klein-Gordon operator permits to separate variables in the wave equation (see (9) below).

Notice that the quantum analogous of H is nothing but $-\frac{1}{2}\nabla^2$.

When V_4 has no more isometries than those corresponding to symmetries of V_3 , it is clear that K is invariant under all isometries of V_4 (unless otherwise specified we consider the connected group).

Beside these obvious symmetries, a question is whether additional isometries may exist in V_4 .

This may happen. For instance let us consider a FRW universe in the usual sense, with maximally symmetric space sections. Any transformation taken from the group of motions in the three-dimensional space corresponds to an isometry of spacetime. We expect that "in the generic case", the metric has no further symmetry.

But if the universe expands in a particular way, it may be a de Sitter manifold; then the group of isometries has ten parameters. The group of motions in V_3 do not exhaust all the isometries of V_4 , but it might be reasonable to require that the Weinberg function be invariant under all of them. Since quantization in de Sitter space has been extensively discussed in the literature [12][14], we shall not dwell further in this direction.

Owing to the importance of isometries in the problem of quantization, it is of interest to make sure that the situation pictured in the above example is in fact exceptional. This point will be clarified in subsection II.3.

II.2 Separation of variables in generalized FRW spacetimes.

For the moment eq.(4) allows one to cast (1) into a remarkably simple form. We use

coordinates such that

$$g^{0i} = g_{i0} = 0 \quad g_{ij} = -B^2 \gamma_{ij}$$

$$\det g_{ij} = B^6 \gamma, \quad \gamma = \det \gamma_{ij}$$

$$g = \det g_{\mu\nu}, \quad \sqrt{|g|} = B^6 \sqrt{\gamma}.$$

It turns out that ∇^2 and therefore the operator in (1) commute with Δ_3 . This fact permits to reduce the wave equation into a one dimensional problem. Solutions to the wave equation exist which are also eigenstates of the three-dimensional Laplacian Δ_3 . In accordance with the usual terminology applied to Friedman universes, we shall call them *mode solutions* of eq. (1).

Let us consider a mode solution, say Φ . For some nonnegative $\lambda \in \text{Spec}(V_3)$ we have

$$\Delta_3 \Phi = -\lambda \Phi \tag{5}$$

and (1) reduces to

$$(\partial^0 \partial_0 + \lambda B^{-2} + \mu) \Phi = 0 \tag{6}$$

with $\partial^0 = B^{-6} \partial_0$. The space variables x^j can be ignored in solving this equation. Since the coefficient of $\partial_0 \partial_0$ in $\partial^0 \partial_0$ never vanishes, equation (6) is always of second order. Here we assume that V_3 is compact and simply connected, which entails [15][16] that λ belongs to the discrete sequence

$$\text{Spec}(V_3) = \{\lambda_0 = 0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \dots \infty\} \tag{7}$$

and that the eigenspace \mathcal{E}_n of $-\Delta_3$ in $C^\infty(V_3)$, associated with the eigenvalue λ_n has finite dimension $r(n)$.

Let \mathcal{S}_n be the space (two-dimensional over complex numbers) of C^∞ functions of t satisfying the equation

$$(\partial^0 \partial_0 + \lambda_n B^{-2} + \mu) f = 0 \tag{8}$$

and let $f_{1,n}(t)$ and $f_{2,n}(t)$ form a basis of \mathcal{S}_n . The general solution of (1) in mode n can be written as

$$\Phi = F_{1,n}(x^j)f_{1,n}(t) + F_{2,n}(x^j)f_{2,n}(t) \quad (9)$$

where $F_{1,n}$ and $F_{2,n}$ belong to \mathcal{E}_n .

Definition: Any basis of \mathcal{S}_n such that $f_{2,n} = f_{1,n}^*$ will be called a canonical basis.

Remark: starting from two independent real solutions of (8) it is always possible to construct a canonical basis. Each such basis permits to define in \mathcal{S}_n the one-dimensional subspaces $\mathcal{S}_n^{(1)}$ and $\mathcal{S}_n^{(2)}$ spanned by $f_{1,n}$ and $f_{2,n}$ respectively.

Definition: A canonical basis $f_{1,n}, f_{2,n}$ is orthonormal when $iW(f_1, f_2) = 1$.

Here the Wronskian $W(f, h) \equiv f\dot{h} - h\dot{f}$ is constant in t provided both f and h are solutions of the same equation (8).

Our terminology is justified by the fact that $-iW(f^*, h)$ is a sesquilinear form of the couple f, h in \mathcal{S}_n .

It is natural to postulate that the projectors Π^\pm commute with K . This property is always implicitly assumed in the literature. It amounts to require that the splitting into positive and negative-frequency subspaces is first performed in each "kinetic-energy shell" (eigenspace common to ∇^2 and Δ_3) and further extended by direct sum. As a result, the only freedom left for defining positive-energy solutions concerns the splitting of $\mathcal{S}_n S$ into $\mathcal{S}_n^{(1)}$ and $\mathcal{S}_n^{(2)}$. One must proceed so in each kinetic-energy shell, but in this section the eigenvalue λ_n is kept fixed. Whenever no confusion is possible, we shall drop the label n referring to it.

In order to discuss the surviving degree of arbitrariness, let us consider two admissible choices respectively characterized by f_1, f_2 and another canonical basis, say f'_1, f'_2 .

The sesquilinear form

$$(\Phi; \Omega) = \int j^\nu(\Phi, \Omega) d\Sigma_\nu \quad (10)$$

associated with the Gordon current

$$j^\nu(\Phi, \Omega) = -i(\Phi^* \nabla^\nu \Omega - \Omega \nabla^\nu \Phi^*) \quad (11)$$

is independent of the spacelike hypersurface Σ for any couple of solutions to (1). For product solutions of the form $\Phi = f(t)F(\xi)$, $\Omega = h(t)H(\xi)$, namely for mode solutions, our choice of time coordinate permits to write this simple formula

$$(\Phi; \Omega) = \int_{\Sigma} j^\nu(\Phi, \Omega) d\Sigma_\nu = -iW(f^*, h) \int F^* H \sqrt{\gamma} d^3\xi \quad (12)$$

The integral in the r.h.s. is the three-dimensional scalar product

$$((F, H)) = \int F^* H \sqrt{\gamma} d^3\xi$$

invariant by the isometries of V_3 endowed with metric γ (*spatial* isometries).

Notice that formula (12) ensures *positivity* of $(\Phi; \Omega)$ provided the basis of \mathcal{S} is suitably normalized. We can always choose our notation such that $-iW(f_2, f_1) = 1$. As seen in a previous work (Propo.3 in ref.[8]) we can assert that $(\Phi; \Phi) \geq 0$ and vanishes only for $\Phi = 0$, when $\Phi \in \mathcal{S}^{(1)} \otimes \mathcal{E}$. Similarly $(\Phi; \Phi) \leq 0$ and vanishes only for $\Phi = 0$, when $\Phi \in \mathcal{S}^{(2)} \otimes \mathcal{E}$.

The *scalar product* is defined as $\langle \Phi, \Psi \rangle = \pm(\Phi; \Psi)$ respectively in $\mathcal{S}^{(1)} \otimes \mathcal{E}$ and $\mathcal{S}^{(2)} \otimes \mathcal{E}$, whereas $\mathcal{S}^{(1)} \otimes \mathcal{E}$ and $\mathcal{S}^{(2)} \otimes \mathcal{E}$ are mutually orthogonal in the scalar product as well as they are in the sesquilinear form (10). In contradistinction to this sesquilinear form, the scalar product defined above is positive definite, hence Hilbertian, for $\dim \mathcal{E}$ is finite. But it crucially depends on the splitting we have performed, that is on the choice of an orthonormal canonical basis in \mathcal{S} .

II.3 Isometries.

Generalized FRW spacetimes have obvious symmetries: It is clear that any isometry of V_3 also leave invariant the spacetime metric. But the question is to investigate if other isometries can occur in V_4 .

Proposition 1.

Except perhaps in the case where the scale factor satisfies a particular law of evolution, all connected isometries of V_4 are induced by those of V_3 .

The proof is purely local; it essentially involves infinitesimal generators. Let X be a Killing vector on V_4 . When it corresponds to a group of motion in V_3 , its component X^0 vanishes. Otherwise, we shall say that X generates a "nontrivial group of isometries". But is this latter case really possible? The answer is *no* in the generic case precisely characterized below.

Calculations are carried out in Appendix, using the cosmic time coordinate, and setting $S = B^2$. It is clear that the vanishing of X^0 is not affected by any rescaling of the form $t = t(\bar{t})$.

Starting from the Killing equation one finds that X^0 necessarily vanishes, except if S satisfies

$$\frac{d^2 S}{d\bar{t}^2} - S^{-1} \left(\frac{dS}{d\bar{t}} \right)^2 = \text{const} \quad (13)$$

In the sequel, we make the convention that a *generic* scale factor *does not* satisfy the above equation.

Proposition 2.

In the generic case, all the connected isometries of V_4 are naturally represented as unitary transformations of $\mathcal{S}^{(1)} \otimes \mathcal{E}$ endowed with the scalar product $\langle \Phi, \Psi \rangle$.

Indeed we know from Proposition 1 that they are provided by spatial symmetries. Let T be such a symmetry. It maps \mathcal{E} onto itself. We shall first define $TF = F(T\xi)$. Then T is extended to $\mathcal{S}^{(1)} \otimes \mathcal{E}$ as follows. Say $\Psi = f_1(t) F(\xi)$. We write

$$T\Psi = f_1(t) F(T\xi) \quad (14)$$

Obviously T maps $\mathcal{S}^{(1)} \otimes \mathcal{E}$ into itself. It is clear that T is unitary in \mathcal{E} endowed with the elliptic scalar product $((F, G))$, obviously invariant under the isometries of V_3 . Then we

find $\langle T\Phi, T\Psi \rangle = \langle \Phi, \Psi \rangle$ with help of formula (12). Since T is supposed to belong to a continuous group it is invertible, which achieve to prove unitarity.

Of course we have the same property with respect to $\mathcal{S}^{(2)} \otimes \mathcal{E}$.

From now on we respectively identify

$$\mathcal{H}_n^{(1)} = \mathcal{S}_n^{(1)} \otimes \mathcal{E}_n, \quad \mathcal{H}_n^{(2)} = \mathcal{S}_n^{(2)} \otimes \mathcal{E}_n$$

as the positive- and negative-frequency subspace in the mode labelled by n . According to (9), the n^{th} space of mode-solutions is

$$\mathcal{H}_n = \mathcal{H}_n^{(1)} \oplus \mathcal{H}_n^{(2)}$$

In the sequel we shall indifferently write \mathcal{H}_n^+ for $\mathcal{H}_n^{(1)}$, etc.

III. POSITIVE-FREQUENCY KERNEL

III.1 Kernel at a given mode.

We turn back to quantum mechanics and consider equation (1) when the metric is of the form (4). Separation of frequencies is supposed to respect kinetic-energy shells, therefore in each shell \mathcal{H}_n we must have the projectors Π_n^\pm associated with the kernels D_n^\pm .

But kinetic energy is kept fixed throughout *this section*; therefore, *the label n referring to a determined eigenvalue λ_n is dropped in intermediate calculations.*

In this section we assume that some canonical orthonormal basis of \mathcal{S} is choosen. Let it be f_1, f_2 with $f_2 = f_1^*$. Being orthonormal it satisfies $-iW(f_2, f_1) = 1$.

It is required that D^+ itself is (possibly in the sense of distributions) a positive-frequency solution of (1). Therefore the kernel $D_n^+(y, x)$, as a function of x , must belong to $\mathcal{S}_n^{(1)} \otimes \mathcal{E}_n$.

In other words, with an obvious notation $y = (u, \eta)$, $x = (t, \xi)$, we can write

$$D_n^+(y, x) = f_1(t) L(y, \xi) \tag{15}$$

where L , as a function of ξ must belong to \mathcal{E} . Naturally, owing to the bi-scalar nature of the kernels, L may additionally depend on y .

Here we take advantage of the fact that $\dim \mathcal{E} = r(n) < \infty$. Let E_1, E_2, \dots, E_r be an orthonormal basis of \mathcal{E} , that is $((E_a, E_b)) = \delta_{ab}$. We can write L as

$$L = w_1 E_1 + \dots w_r E_r \quad (16)$$

with y -depending coefficients. Hence

$$D^+(y, x) = f_1(t) \sum w_b E_b(\xi) \quad (17)$$

Consider any positive-energy solution Ψ in mode n . It has a development

$$\Psi = \sum f_1(t) \psi_a E_a(\xi) \quad (18)$$

where ψ_1, \dots, ψ_r are complex constants. According to (2) we must have

$$((D_n^+)_y; \Psi) = \Psi(y) \quad (19)$$

where the notation $(D_n^+)_y$ indicates that the sesquilinear form is calculated by integrating over x , and that D_n^+ additionally depend on y . According to the developments (17)(18) we find

$$(D_y^+; \Psi) = \sum w_a^* \psi_b (f_1(t) E_a(\xi); f_1(t) E_b(\xi))$$

Use (12) and remember that $-iW(f_1, f_1) = 1$, we get

$$(D_y^+; \Psi) = \sum w_a^* \psi_b ((E_a; E_b))$$

$$(D_y^+; \Psi) = \sum w_a^* \psi_a$$

This quantity must coincide with $\Psi(y)$, given by (18) where t, ξ are replaced by u, η . In all this the coefficients ψ_a are arbitrary. Therefore it is necessary that

$$w_a^*(y) = f_1(u) E_a(\eta)$$

and the only possibility is

$$D^+(y, x) = f_1^*(u) f_1(t) \sum E_a^*(\eta) E_a(\xi) \quad (20)$$

In other words, $D^+ = f_2(u)f_1(t) \Gamma(\eta, \xi)$, if we set

$$\Gamma(\eta, \xi) = \sum E_a^*(\eta)E_a(\xi) \quad (21)$$

It is not difficult to check that expression (20) of D_n^+ actually implies eq.(2) as it should. Moreover the formula (21) is invariant under changes of orthonormal basis inside \mathcal{E} . Therefore Γ is intrinsically defined; it is a purely spatial quantity, independent of the splitting performed in \mathcal{S} .

Notice that Γ is a reproducing kernel in V_3 , for we check $((\Gamma_\eta, F)) = F(\eta)$ for all $F(\xi) \in C^\infty(V_3)$. Moreover, using a basis of *real functions* in \mathcal{E} , we see that $\Gamma^* = \Gamma$.

In contradistinction the factor $f_1^*(u)f_1(t)$ depends on the choice of a basis in \mathcal{S} . For usual FRW spacetimes (V_3 is of constant curvature), the explicit form of E_1, \dots, E_r can be found in the literature; see references [15][16].

As expected we check that $D_n^+(y, x)^* = D_n^+(x, y)$.

Looking for isometric invariance we can claim

Γ is invariant under all the isometries of V_3 .

Proof:

Let T be an isometry of the metric γ_{ij} , we also denote $TF = F(T\xi)$. Since transformation T leaves invariant the three-dimensional Laplacian, the eigenspaces \mathcal{E} are globally invariant. Moreover T leaves invariant the three-dimensional scalar product, in other words

$$((TF, TG)) = ((F, G))$$

for $F, G \in C^\infty(V_3)$. This is true in particular when F and $G \in \mathcal{E}$. Since T is taken from a continuous group it is inversible; it follows that T is a unitary transformation of \mathcal{E} endowed with its scalar product above. Thus $E_1(T\xi), \dots, E_r(T\xi)$ is another basis of \mathcal{E} . We can write

$$E_a(T\xi) = \sum T_{ab}E_b(\xi)$$

hence

$$E^*(T\eta) = \sum T_{ac}^*E_c^*(\eta)$$

Now consider

$$\begin{aligned}\Gamma(T\eta, T\xi) &= \sum E_a^*(T\eta) E_a(T\xi) \\ \Gamma(T\eta, T\xi) &= \sum T_{ac}^* E_c^*(\eta) T_{ab} E_b(\xi)\end{aligned}$$

But $T_{ac}^* = (T^{-1})_{ca}$ because of unitarity. Thus $\Gamma(T\eta, T\xi) = \Gamma(\eta, \xi)$.

Now return to (17). Transformations in V_3 do not affect \mathcal{S} and leave f_1, f_2 unchanged. Proposition 1 ensures that any connected isometry of V_4 stems from an isometry of V_3 and therefore D^+ is invariant as well as Γ ; restoring the mode label we shall write

$$D_n^+(Ty, Tx) = D_n^+(y, x) \tag{22}$$

We shall summarize:

The only kernel D_n^+ solution of the wave equation, eigenfunction of $-\Delta$ for eigenvalue λ_n and satisfying (2) where Ψ is a mode solution, is given by (17). In the generic case, it is isometrically invariant.

Of course, in the exceptional case it remains at least invariant under the subgroup of purely *spatial* isometries.

Now the next point consists in showing that two possible candidates for D_n^+ are necessarily connected by a unitary transformation which respects isometries.

By (20)(21) it is clear that any alternative candidate for D^+ is necessarily of the form

$$D'^+ = f_1'^*(u) f_1'(t) \Gamma(\eta, \xi)$$

where $f_1', f_2' = f_1'^*$ form another normed canonical basis of \mathcal{S} . We can write

$$f_1' = \alpha f_1 + \beta f_2$$

with $\alpha, \beta \in \mathbf{C}$. Orthonormality of the new basis reads $\alpha\alpha^* - \beta\beta^* = 1$ as is well known. This change of basis can be seen also as a map of \mathcal{S} onto itself (Bogoliubov transformations).

Restricted as a mapping of $\mathcal{S}^{(1)}$ onto $\mathcal{S}'^{(1)}$, it is unitary, say $f'_1 = U f_1$. Then U is extended as a unitary map from $\mathcal{S}^{(1)} \otimes \mathcal{E}$ to $\mathcal{S}'^{(1)} \otimes \mathcal{E}$, according to the rule $U(fF) = (Uf)F$. Now, noticing that $D_y^+ \in \mathcal{S}^{(1)} \otimes \mathcal{E}$ as a function of x , we can write *in each mode*

$$(D_n'^+)_y = U_n(D_n^+)_y \quad (23)$$

It is a trivial point that spatial isometries, understood as transformations acting in $\mathcal{E}_n \otimes \mathcal{E}_n$, commute with U_n .

Up to now we have proceeded with kinetic energy fixed. The last step consists in summing over all the possible values of the integer n labelling the eigenvalue λ .

III.2 Sum over modes

For each mode, $\mathcal{H}_n^{(1)}$ is equipped with the metric $\langle \Phi, \Phi \rangle$ and $\mathcal{H}_n^{(2)}$ with the opposite.

Being of finite dimension $\mathcal{H}_n^{(1)}$ and $\mathcal{H}_n^{(2)}$ are complete as Hilbert spaces.

The direct sums $\mathcal{K}^{(1)} = \bigoplus \mathcal{H}_n^{(1)}$ and $\mathcal{K}^{(2)} = \bigoplus \mathcal{H}_n^{(2)}$ undergo a decomposition which generalizes the separation of frequencies available in static spacetimes. Obviously $\mathcal{K}^{(1)} \oplus \mathcal{K}^{(2)} = \bigoplus \mathcal{H}_n$. The infinite sum $\Phi = \sum_0^\infty \Phi_n$ where $\Phi_n \in \mathcal{H}_n$, is in \mathcal{K} only when $\sum \langle \Phi_n, \Phi_n \rangle >^2$ is finite, but Φ always exists in the distributional sense, if we define as *test functions* the sums $\Psi = \sum \Psi_n$ having an arbitrary but *finite* number of terms in \mathcal{H}_n (terminating sums). Of course $D^+ = \sum D_n^+$ exists as a distribution in the above sense (in both arguments).

When $U_1, U_2, \dots, U_n, \dots$ is a sequence such that every U_n acts unitarily in \mathcal{H}_n , define

$$(\sum U_n)(\sum \Phi_n) = \sum U_n \Phi_n \quad (24)$$

It is clear that $U = \sum U_n$ is a unitary operator in \mathcal{K} . Moreover U maps into itself the space of terminating sequences therefore *it can be extended to distributions*.

The above definition applies to bi-scalars with respect to both arguments, say

$$(U)_x(U)_y G = \sum (U_n)_x(U_n)_y G_n(x, y) \quad (25)$$

where $G = \sum G_n$ may be a distribution whereas $G_n \in (\mathcal{H})_x \otimes (\mathcal{H})_y$.

Let us now take for each U_n the transformation T_n induced in \mathcal{H}_n by a connected isometry of V_4 . Isometric transformations are extended to the full space of solutions by equation (24). Again this U maps into itself the space of terminating sums, so it can be applied to distributions.

It follows from (22) and the above formula that D^+ is invariant under $(T)_x(T)_y$.

The most general transformation relating two different definitions of $\mathcal{K}^{(1)}$ is given by $U = \sum U_n$ where now each U_n is an arbitrary Bogoliubov transformation at mode n .

Remark: a high energy cut-off (imposing that all Φ_n vanish for n greater than some fixed integer) would respect isometric invariance.

We summarize this section in the following statement:

The positive-frequency kernel is given by

$$D^+(x, y) = \sum_{n=0}^{\infty} D_n^+(x, y) \quad (26)$$

where D_n^+ takes on the form (20) in each kinetic-energy shell.

Proposition 3

The only kernel solution of the wave equation, mode-wise defined and satisfying (2) is given by the above formula. It is defined up to a unitary transformation $U = \bigoplus U_n$ where each U_n is an arbitrary Bogolubov transformation in mode n .

In the generic case, D^+ is invariant under the connected isometries of V_4 .

”Mode-wise defined” means that equation (2) is satisfied mode by mode, and refers to the existence of D_n^+ for all n .

IV. CONCLUSION

The splitting of the space of solutions into positive-frequency and negative-frequency subspaces has been carried out separately in each kinetic-energy shell, and further extended to the whole space of solutions. As expected from previous works on the subject, the only ambiguity (for a given value of kinetic energy) arises in the reduced space of solutions, which is

two-dimensional, and corresponds to the possibility of performing an arbitrary Bogoliubov transformation. Indeed the expansion (20) of D^+ in terms of the eigenfunctions of Δ_3 still depends, for each mode, on the choice of a solution to the reduced (one-dimensional) wave equation, restricted only by the Wronskian condition.

As a result, there are infinitely many possible definitions of the one-particle space.

However the total kernel D^+ obtained by summing over modes remains manifestly invariant under the isometries of space sections. Having checked that, beside an exceptional case, V_4 has generally no more connected isometries than those inherited from its space sections, we are in a position to claim that the connected group of spacetime isometries leaves the positive-frequency kernel invariant, irrespective of the choice made among all possible candidates. Notice that the (unitary) transformation connecting two possible definitions of D^+ commutes with connected isometries.

When V_3 enjoys discrete symmetries, which is the case in conventional FRW universes (V_3 being the three-dimensional sphere), it is a mere exercise to check that these symmetries leave D_n^+ invariant and that they act unitarily in each \mathcal{H}_n^+ . Extension to D^+ and \mathcal{K}^+ is straightforward. But of course these discrete *spatial* symmetries may fail to exhaust the group of discrete isometries of V_4 . There is no statement analogous to Proposition 1 for discrete transformations. In fact there are models where V_4 is time-reversal invariant whereas, like in Minkowski space, time reflexion transforms D^+ into a negative-frequency solution of the wave equation.

The technical results presented here are formulated in a self-contained manner within the single-particle sector. It is noteworthy that the wronskian condition arises as a normalization condition without need to invoke, at this stage, the canonical Hamiltonian formalism for fields.

Next step will be a Fock space construction and the definition of field operators. Such a definition essentially involves nothing but the creation and annihilation operators associated with D^+ . We can anticipate that the invariance of D^+ will be naturally extended to the

whole theory of free fields; this point is important because isometric invariance plays in curved spacetime the same role as Poincaré invariance in special Relativity.

We have left aside the special cases where extra isometries could exist. This case is not empty as we can see by looking at the de Sitter space, exceptional in many respects. Clearly our approach is based upon the existence of a preferred family of space sections, so it is bound to break down in the de Sitter case. Fortunately the existence of invariant kernels in this particular spacetime are well-known [12][14].

The possibility of connecting two different definitions of the one-particle space by a unitary transformation was known previously [16][4]. But here we have additionally proved that (in the generic case) the connected group of spacetime isometries acts unitarily in $\mathcal{H}^+ = \bigoplus_n \mathcal{H}_n^+$, whatever is the choice of admissible basis made in each \mathcal{S}_n .

This fact strongly suggests that we treat all admissible choices on equal footing. In physical terms, it means that all possible definitions of the positive-frequency one-particle states could correspond to equivalent representations of the same physics. In other words, the tremendous arbitrariness involved in the formalism is physically irrelevant, and can be seen as a sort of gauge freedom.

This interpretation must be understood as an attempt to give a definition of particle *as much as possible* independent of the observer. It could be easily generalized to several other classes of spacetimes.

Of course the operational meaning of this picture remains to be investigated (for instance the presence of a horizon could perhaps result in some limitations). In our opinion, this task should be undertaken only *after* Fock space construction is achieved.

When statically bounded expansion is considered, it becomes clear that our definition of a particle departs from the popular scheme of "in and out vacua". Insofar as detectors and observers should be invoked at this stage, our point of view amounts to think of a unique class of observers, rather than of two classes of observers respectively submitted to "in"

and "out" asymptotic conditions.

In contradistinction, there is no conceptual discrepancy between the interpretation proposed here and various efforts made in the past [5] and recently [8][17] in order to exhibit, under some circumstances, a unique, distinguished, "definition of the vacuum". Indeed, inasmuch as a distinguished definition can be actually considered as satisfactory, it becomes a matter of taste to utilize this representation rather than any other one equivalent to it.

The author is indebted to J.Renaud and P.Teyssandier for discussions about de Sitter spacetimes.

APPENDIX 1

It is convenient to define $S = B^2$ and to utilize the cosmic time \bar{t} . In this section, overlines will be provisionally dropped for typographical simplicity.

The calculation presented below is affected by the dimensionality of spacetime. Although we are mainly interested by the case of V_4 , applications in different contexts may be of interest. Therefore, in this Appendix, let $V_{q+1} = \mathbf{R} \times V_q$ be the spacetime manifold. Here i, j, k run from 1 to q , and Greek labels from 0 to q . Parenthesis for indices means symmetrization.

The argument is purely local; compactness of V_q plays no role.

Using the cosmic time coordinate, it is possible to cast the metric into the following form

$$ds^2 = dt^2 - S(t)\gamma_{ij}dx^i dx^j \quad (A1)$$

The point $\xi \in V_q$ has coordinates x^i . In these coordinates, $g_{00} = 1$, $g_{0i} = 0$ and $g_{ij} = -S(t)\gamma_{ij}(\xi)$. Coordinates in $V - q$ are arbitrary until we need to specify them. Beware that here a dot means differentiation with respect to the comoving time.

For a Killing vector X of V_{q+1} , we are interested in developing the equation

$$\nabla_\mu X_\nu + \nabla_\nu X_\mu = 0 \quad (A2)$$

The question is whether X^0 may be different from zero. Christoffel symbols are as follows [18]

$$\Gamma_0^0{}_0 = \Gamma_0^0{}_i = \Gamma_0^i{}_0 = 0$$

$$\Gamma_i^0{}_j = \frac{1}{2} \partial_0 g_{ij} \quad (A3)$$

$$\Gamma_0^i{}_j = \frac{1}{2} g^{ik} \partial_0 g_{kj} \quad (A4)$$

But $g^{ij} = -S^{-1} \gamma^{ij}$, hence

$$g^{ik} \partial_0 g_{kj} = \frac{\dot{S}}{S} \delta_j^i$$

and finally

$$\Gamma_0^i{}_j = \frac{\dot{S}}{2S} \delta_j^i \quad (A5)$$

The remaining components of the affinity are

$$\Gamma_i^k{}_j = (\tilde{\Gamma})_i^k{}_j \quad (A6)$$

where $\tilde{\Gamma}$ is the affine connection for the time-dependent tensor g_{ij} defined on V_3 . Elementary manipulations show that

$$\tilde{\Gamma} = \bar{\Gamma} \quad (A7)$$

where $\bar{\Gamma}$ is the affinity of the three-metric γ of V_3 .

Let us now express the contents of (A2). It is clear that $\nabla_0 X_0 = 0$. Hence

$$\partial_0 X_0 = 0$$

Thus $X_0 = X^0$ depends only on ξ . Now, on the one hand we have

$$\nabla_0 X_i = \partial_0 X_i - \Gamma_0^j{}_i X_j$$

From (A5) we find

$$\nabla_0 X_i = \partial_0 X_i - \frac{\dot{S}}{2S} X_i \quad (A8)$$

On the other hand we have

$$\nabla_i X_0 = \partial_i X_0 - \Gamma_i^j{}_0 X_j$$

According to (A5) we get

$$\nabla_i X_0 = \partial_i X_0 - \frac{\dot{S}}{2S} X_i \quad (A9)$$

The sum of (A8)(A9) and condition that $\nabla_{(i} X_{j)}$ vanishes give

$$\partial_0 X_i + \partial_i X_0 = \frac{\dot{S}}{S} X_i \quad (A10)$$

Then, we must express that $\nabla_{(i} X_{j)}$ vanishes.

$$\nabla_i X_j = \partial_i X_j - \Gamma_i^k{}_j X_k - \Gamma_i^0{}_j X_0$$

By (A6)(A7)(A3)

$$\nabla_i X_j = \partial_i X_j - \bar{\Gamma}_i^k{}_j X_k + \frac{1}{2}(\partial_0 g_{ij}) X_0$$

where $\partial_0 g_{ij} = -\dot{S}\gamma_{ij}$.

$$\nabla_i X_j = \partial_i X_j - \bar{\Gamma}_i^k{}_j X_k - \frac{\dot{S}}{2}\gamma_{ij} X_0 \quad (A11)$$

So we must write

$$\partial_i X_j + \partial_j X_i - 2\bar{\Gamma}_i^k{}_j X_k - \dot{S}\gamma_{ij} X_0 + 0 \quad (A12)$$

Contracted multiplication of (A12) by γ^{ij} yields

$$2\gamma^{ij}\partial_i X_j - 2\gamma^{ij}\bar{\Gamma}_i^k{}_j X_k - q\dot{S}X_0 = 0$$

(since space sections are q -dimensional). So far the coordinates in V_q have been left arbitrary. Using now harmonic coordinates in V_q , we cancel the second term in the above formula. Hence $2\gamma^{ij}\partial_i X_j = q\dot{S}X_0$ and after differentiation with respect to x^0 ,

$$2\gamma^{ij}\partial_0\partial_i X_j = q\ddot{S}X_0 \quad (A13)$$

Now, differentiating (A10) with respect to the j^{th} coordinate we obtain

$$\partial_0\partial_j X_i + \partial_i\partial_j X_0 = \frac{\dot{S}}{S}\partial_j X_i$$

Since $\partial_0 \gamma^{ij} = 0$, we notice that

$$\partial_0(\gamma^{ij} \partial_i X_j) = \gamma^{ij} \partial_0 \partial_i X_j$$

hence the contraction

$$\partial_0(\gamma^{ij} \partial_i X_j) + \gamma^{ij} \partial_j \partial_i X_0 = \frac{\dot{S}}{S} \gamma^{ij} \partial_j X_i \quad (A14)$$

Insert (A13) into (A14); we finally get

$$(\ddot{S} - \frac{\dot{S}^2}{S})X_0 + \frac{2}{q} \gamma^{ij} \partial_j \partial_i X_0 = 0$$

As the second term doesnot depend on time, it is necessary that

$$\ddot{S} - \frac{\dot{S}^2}{S} = \text{const} \quad (A15)$$

It is noteworthy that this condition is independent of the dimension. For a flat and open V_q , the vanishing of the constant in the r.h.s. of (A15) corresponds to the steady-state universe $S = \text{const. } e^{lt}$, with Hubble constant l , which covers a half of the de Sitter manifold.

For q -spherical space sections (the case we are more specially concerned with in this paper) we have $\ddot{S} - \dot{S}^2/S = 2$, satisfied by the de Sitter universe, $S = \cosh^2(t/l)$.

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In the particular example of a Friedman universe in the usual sense (flat space sections), the conservation of kinetic energy may be related with the constancy of the space components of momentum, which stems from space translation invariance. But in general kinetic energy is conserved irrespective of a possible isometry group; it may happen that V_3 has no isometry at all, but the form (4) of the metric ensures that K is conserved anyway.
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